

Improved Evaluation of the Hadronic Vacuum Polarization Contributions to Muon $g - 2$ and $\bar{\alpha}_{\text{QED}}(M_Z)$ Using High Order QCD Calculations*

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Abstract

We use recently evaluated radiative and nonperturbative corrections to production of heavy quarks by a vector current to give very precise theoretical calculations of the high energy ($t^{1/2} \geq \sqrt{2}$ GeV) imaginary part of the photon vacuum polarization function, $\text{Im}\Pi(t)$. This allows us to improve the corresponding contributions to the muon (or any other lepton) $g - 2$ anomaly and to the running QED constant on the Z , $\bar{\alpha}_{\text{QED}}(M_Z)$. This decreases the error in the evaluations by a factor between two and six for the high energy contribution, and by some 50% for the overall result. We find for the hadronic contributions $a_h = 6993.4 \pm 110.0 \times 10^{-11}$ and $\Delta\alpha_h = 272.59 \pm 4.09 \times 10^{-4}$.
14.40.Gx, 12.38.Bx, 12.38.Lg, 13.20.Gd

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I. INTRODUCTION

There has been recently a renewed interest ^[1] on two basic quantities, both related to the photon vacuum polarization: the hadronic contribution to the anomalous moment of the muon, a_μ , and the running QED charge $\bar{\alpha}$ defined at the Z particle pole. In what respects the first, the reason is the advanced status of the experiment planned at Brookhaven which should improve the present accuracy by more than one order of magnitude; in what respects the second, because the quantity $\bar{\alpha}_{QED}(M_Z^2)$ plays a leading role in precision determination of electroweak parameters.

In this paper we consider the contributions of the hadronic part of the photon vacuum polarization tensor to a_μ and $\bar{\alpha}_{QED}(M_Z^2)$, that we write respectively as

$$a_\mu^{(2)}(\text{had}) \quad , \quad \Delta\bar{\alpha}_{QED}^{(2)}(\text{had} , M_Z^2) . \quad (1.1)$$

These quantities may be expressed in terms of the photon hadron vacuum polarization function Π_h . One can then write a dispersion relation for this function in such a way that the experimentally accessible quantity

$$R(t) = \frac{\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \quad (1.2)$$

appears instead of Π_h :

$$a_h \equiv a_\mu^{(2)}(\text{had}) = \int_{4m_\pi^2}^{\infty} dt R(t) K(t) \quad , \quad (1.3)$$

$$\Delta\alpha_h \equiv \Delta\bar{\alpha}_{QED}^{(2)}(\text{had} , M_Z^2) = -\frac{\alpha_{QED} M_Z^2}{3\pi} \int_{4m_\pi^2}^{\infty} dt \frac{R(t)}{t(t - M_Z^2)} . \quad (1.4)$$

Here,

$$K(t) = \frac{\alpha_{QED}}{3\pi^2 t} \int_0^1 dz \frac{z^2(1-z)}{z^2 + (1-z)t/m_\mu^2} . \quad (1.5)$$

The reason why we think we can improve on existing estimates of a_h , $\Delta\alpha_h$ is that we may use the very reliable *theoretical* QCD calculations¹ that have been extended with

¹The fact that QCD calculations are more precise than the use of experimental data was already noted in e.g. Ref. [2].

great precision to regions previously inaccessible; in particular ^[3] to the regions right above threshold for heavy quark production ($c\bar{c}$, $b\bar{b}$) as well as the very low ($t^{1/2} \sim 1.2 \text{ GeV}$) energies. The extensions have been possible because of the high orders attained by the QCD calculations ^[4] and the use of the precisely known value of α_s on the τ mass ^[5],

$$\alpha_s(m_\tau^2) = 0.33 \pm 0.05 \quad . \quad (1.6)$$

Likewise, the use of the value ^[6] of α_s at the Z ,

$$\alpha_s(M_Z^2) = 0.119 \pm 0.003 \quad , \quad (1.7)$$

allows us to reduce still further the uncertainties in the high energy regions.

II. CONTRIBUTION FROM THE REGION $t^{1/2} < 1.1 \text{ GeV}$

In the region $t^{1/2} < 1.1 \text{ GeV}$, perturbative QCD is clearly invalid. One thus has to rely on experiment, supplemented by old-fashioned hadron theory. The region may be further split into the ρ resonance region (say, $t^{1/2} \leq 0.8 \text{ GeV}$) and the rest. For a_h we have,

$$a_h(t^{1/2} < 0.8) = (4821 \pm 24 \pm 27) \times 10^{-11} + (25 \pm 3) \times 10^{-11} \quad . \quad (2.1)$$

Here the second term in the r.h.s. is the $\omega - \rho$ interference contribution. The first error is statistical; the second (when given) will be the systematic one. Eq. (2.1) presents the value reported in Ref. [2]; other authors give compatible estimates. We will improve on (2.1) slightly later on.

To this one has to add the contribution of the region $0.8 < t^{1/2} \leq 1.1$ which gives, using experimental data only,

$$a_h(0.8 < t^{1/2} \leq 1.1) = (1100 \pm 94) \times 10^{-11} \quad . \quad (2.2)$$

This region, about which little can be done at present, presents the largest source of error. Combining Eqs. (2.1) and (2.2) we have,

$$a_h(t^{1/2} \leq 1.1 \text{ GeV}) = (5947 \pm 97 \pm 27) \times 10^{-11} . \quad (2.3)$$

For $\Delta\alpha_h$, we have, taking the analysis of the first article in Ref. [1],

$$\Delta\alpha_h(t^{1/2} < 1.1 \text{ GeV}) = (35 \pm 0.3 \pm 0.7) \times 10^{-4} . \quad (2.4)$$

The contribution of the region $1.1 < t^{1/2} < 2 \text{ GeV}$ is the topic of next section.

III. CONTRIBUTION FROM THE REGION $1.1 < t^{1/2} < 2 \text{ GeV}$

From a theoretical point of view, the contribution of this region presents a challenge, because one cannot use perturbative QCD reliably, and also because old-fashioned hadron theory does not describe very well the average experimental data points. In this section, we explain how one can combine these two theories and give reliable *theoretical* estimates, which can then be compared with previous estimates based on *experimental* data only ^[1]:

$$a_h^{\text{exp}}(1.1 \leq t^{1/2} < \sqrt{2} \text{ GeV}) = (278 \pm 25) \times 10^{-11} , \quad (3.1)$$

$$\Delta\alpha_h^{\text{exp}}(1.1 < t^{1/2} < \sqrt{2} \text{ GeV}) = (13 \pm 0.15 \pm 0.8) \times 10^{-4} . \quad (3.2)$$

Let us first estimate the contribution of this region to $R(t)$ using perturbative QCD only: $R(t)$ may be split as a sum over light quark flavors:

$$R(t) = \sum_{q=u,d,s} R_q(t) , \quad (3.3)$$

and one may use the very precise high energy result for $R_q(t)$:

$$\begin{aligned} R_q^{\text{h.e.}}(t) \underset{\bar{v} \rightarrow 1}{=} N_c Q_q^2 \left\{ 1 - \frac{3}{2}(1 - \bar{v})^2 + \frac{1}{2}(1 - \bar{v})^3 + \left[\frac{3}{4} + \frac{9}{2}(1 - \bar{v}) \right] \frac{C_F \alpha_s}{\pi} \right. \\ \left. + \left[\frac{9}{2} \left(\ln \frac{2}{1 - \bar{v}} - \frac{3}{8} \right) (1 - \bar{v})^2 \right] \frac{C_F \alpha_s}{\pi} + r_2 \left(\frac{\alpha_s}{\pi} \right)^2 + \tilde{r}_3 \left(\frac{\alpha_s}{\pi} \right)^3 \right. \\ \left. + \frac{9}{2}(1 - \bar{v}) C_F \left[8.7 \left(\frac{\alpha_s}{\pi} \right)^2 + 45.3 \left(\frac{\alpha_s}{\pi} \right)^3 \right] \right\} . \end{aligned} \quad (3.4)$$

Here,

$$\begin{aligned}\bar{v} &\equiv \sqrt{1 - 4\bar{m}^2(t)/t} \quad , \quad r_2 = 1.986 - 0.115 n_f \quad , \quad C_F = 4/3 \quad , \\ \tilde{r}_3 &= -6.637 - 1.2 n_f - 0.005 n_f^2 - 1.24 \left(\sum_f Q_f \right)^2 \quad ,\end{aligned}\tag{3.5}$$

and $\bar{m}(t)$ and $\alpha_s \equiv \alpha_s(t)$ are the running mass and coupling constant which we may take to two loops ^[7]. The *justification* of this approximation lies in the fact that the light quarks are relativistic at these energies. Numerically, we take $\bar{m}_u = \bar{m}_d = 0$, and use ^[7]

$$\bar{m}_s(1 \text{ GeV}^2) = 0.19 \text{ GeV} \quad .\tag{3.6}$$

The result $R_{QCD}(t)$ obtained in this approximation is plotted in Fig. 1. together with some experimental points. Although the experimental errors are large, it is clear that our curve R_{QCD} will not give a reliable estimate. Before we explain how this can be improved, let us first describe another approximation for $R(t)$ based on old-fashioned hadron theory.

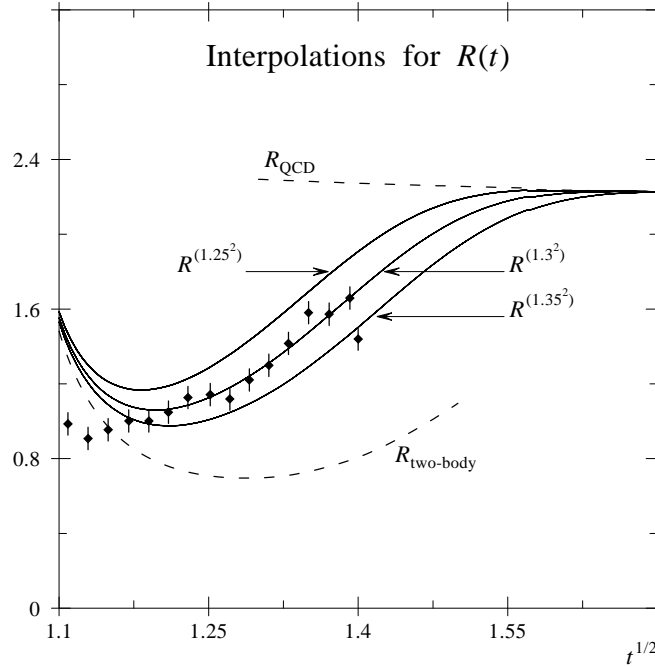


Fig. 1.

To do this, we evaluate the contribution of the individual channels, in the quasi-two-body approximation. Thus, we have the two-body channels $\pi^+\pi^-$, K^+K^- , $K^0\bar{K}^0$, and the quasi-two-body channels $\rho^+\pi^-$, $\rho^-\pi^+$, $\rho^0\pi^0$, $\rho^0\eta$, $\omega\pi^0$, $\phi^0\pi^0$, $\phi^0\eta$, $K^{*0}K^0$, $K^{*+}K^-$, $K^{*-}K^+$.

The contribution of the first is estimated using the corresponding form factors, ρ or ϕ dominated: we obtain

$$\Delta R(t)^{\text{resonances}} = |F_\pi(t)|^2 \frac{1}{4} \left(1 - \frac{4m_\pi^2}{t}\right) + |F_K(t)|^2 \frac{1}{4} \left(1 - \frac{4m_K^2}{t}\right) , \quad (3.7)$$

where

$$\begin{aligned} F_\pi(t) &= \frac{m_\rho^2 + m_\rho m_\pi \xi_0}{m_\rho^2 - t - i m_\rho (t/4 - m_\pi^2)^{3/2} \xi_0} , & \xi_0 &\equiv \frac{\Gamma_\rho}{(m_\rho^2/4 - m_\pi^2)^{3/2}} ; \\ F_K(t) &= \frac{m_\phi^2 + m_\phi m_K \xi_1}{m_\phi^2 - t - i m_\phi (t/4 - m_K^2)^{3/2} \xi_1} , & \xi_1 &\equiv \frac{\Gamma_\phi}{(m_\phi^2/4 - m_K^2)^{3/2}} . \end{aligned}$$

Note that we have combined the contribution coming from both $K^+ K^-$ and $K^0 \bar{K}^0$ into the function F_K .

The quasi-two-body channels contribution to $R(t)$ is estimated in the vector-meson dominance approximation:

$$\Delta R(t)^{\text{quasi-2-body}} = \sum_{V,P=\rho+\pi^-, \dots} \frac{\Gamma(\gamma^* \rightarrow VP)}{\Gamma(\gamma^* \rightarrow \mu^+ \mu^-)} . \quad (3.8)$$

Using the phenomenological interactions

$$\mathcal{L}_{VP} = g_{VP} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} V_{\alpha\beta} \phi_P , \quad (3.9)$$

one can relate this to the radiative decay widths,

$$\Delta R(t) = \sum_{V,P=\rho+\pi^-, \dots} \frac{3 m_V^3 \Gamma(V \rightarrow \gamma P)}{\alpha_{QED} (m_V^2 - m_P^2)^3} \frac{[t - (m_V + m_P)^2]^{3/2} [t - (m_V - m_P)^2]^{3/2}}{t^{1/2} (t - 4m_\mu^2)^{3/2}} . \quad (3.10)$$

The decay widths are taken from experiment. The result $R(t)_{\text{two-body}}$ we obtain in this approximation is also plotted in Fig. 1. It unfortunately does not represent an average of the experimental points. To bring these approximations closer to experiment, we construct the following interpolation curve which reproduces both evaluations in their ranges of validity: we note that QCD works at the higher energies, and the old-fashioned hadron evaluation at the lower energies. Then,

$$\begin{aligned} R^{(t_0)}(t) &\equiv R(t)_{\text{two-body}} \exp(-t^5/\sqrt{5} t_0^5) + R_{QCD}(t) [1 - \exp(-t^5/\sqrt{5} t_0^5)] , \\ t_0^{1/2} &= 1.25 , 1.3 , 1.35 \text{ GeV} . \end{aligned} \quad (3.11)$$

The point t_0 is chosen to fit experiment. These curves are drawn in Fig. 1 (solid curves) and represent a significant improvement from what we had previously. The numerical results obtained with these interpolation functions are:

$$a_h(1.1 < t^{1/2} \leq \sqrt{2} \text{ GeV}) = (265 \pm 22) \times 10^{-11} , \quad (3.12)$$

$$\Delta\alpha_h(1.1 < t^{1/2} \leq \sqrt{2} \text{ GeV}) = (15.3 \pm 1.4) \times 10^{-4} . \quad (3.13)$$

The central values have been obtained using $R(t_0^{1/2}=1.3 \text{ GeV})$, and the systematic errors have been estimated from the difference $\left| R(t_0^{1/2}=1.3 \text{ GeV}) - R(t_0^{1/2}=1.35 \text{ GeV}) \right|$. (These errors are in our opinion over-estimated and should in principle be reduced. We will however not do that here). Comparing the numbers given in Eqs. (3.12, 3.13) with those given in Eqs. (3.1, 3.2) we see they agree, within errors. We however believe our evaluation to be more reliable than those estimates which were obtained using experimental data only, because of the known existence of large systematic errors of the last.

IV. CONTRIBUTION FROM THE REGION $t > 2 \text{ GeV}^2$

For $t > 2 \text{ GeV}^2$, we distinguish two different contributions. The first is due to the J/ψ and Υ bound states and the results can be taken from the first article of Ref. [1]:

$$a_h(J/\psi) = (86 \pm 4.1 \pm 4) \times 10^{-11} , \quad (4.1)$$

$$a_h(\Upsilon) = (1 \pm 0 \pm 0.1) \times 10^{-11} , \quad (4.2)$$

$$\Delta\alpha_h(J/\psi) = (11.34 \pm 0.55 \pm 0.61) \times 10^{-4} , \quad (4.3)$$

$$\Delta\alpha_h(\Upsilon) = (1.18 \pm 0.05 \pm 0.06) \times 10^{-4} . \quad (4.4)$$

The second contribution comes from the continuum regions, i.e. the regions above $q\bar{q}$ thresholds. As before, we split $R(t)$ as

$$R(t) = \sum_{q=u,d,s,c,b,t} R_q(t) . \quad (4.5)$$

For the light quarks u , d , s , only the high energy region needs to be considered; the corresponding expression for R_q was given in Eq. (3.4). For the *heavy* quarks, c , b , t , we also need the value of R_q for small and intermediate velocities. At low energy, the value of R_q is known with great precision [3]:

$$\begin{aligned}
R_q^{\text{l.e.}}(t) \underset{v \rightarrow 0}{=} N_c Q_q^2 & \left\{ \frac{v(3-v^2)}{2} + \left(-\frac{6v}{\pi} + \frac{3\pi v^2}{4} \right) C_F \tilde{\alpha}_s \right\} \left(1 - \frac{2\pi \langle \alpha_s G^2 \rangle}{192 m^4 v^6} \right) \\
& \times [1 + 2c_0(t)] \frac{\pi C_F \tilde{\alpha}_s / v}{1 - e^{-\pi C_F \tilde{\alpha}_s / v}} , \\
v \equiv \sqrt{1 - 4m^2/t} \quad , \quad \tilde{\alpha}_s &= \alpha_s(t) \left[1 + \frac{a_1 + \gamma_E \beta_0 / 2}{\pi} \alpha_s(t) \right] , \\
a_1 = \frac{93 - 10n_f}{36} \quad , \quad \beta_0 &= 11 - \frac{2}{3}n_f \quad ,
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
c_0(t) \underset{ka \ll 1}{=} \frac{\beta_0 \alpha_s(t)}{4\pi} & \left[\ln \frac{t^{1/2} a}{2} - 1 - 2\gamma_E + \frac{(ka)^2}{12} + \frac{(ka)^4}{40} + \dots \right] , \\
k \equiv mv \quad , \quad a \equiv 2/mC_F \tilde{\alpha}_s(t) \quad .
\end{aligned}$$

An exact expression for $c_0(t)$ (which represents the radiative corrections) may be found in Ref. [3]. Eq. (4.6) includes also nonperturbative corrections with $\langle \alpha_s G^2 \rangle = 0.042 \pm 0.020 \text{ GeV}^4$, and the evaluation is valid until these are of order unity, *i.e.*, down to a critical velocity

$$v_{\text{crit}} \sim \left(\frac{2\pi \langle \alpha_s G^2 \rangle}{192 \beta_0 m^4} \right)^{1/6} . \tag{4.7}$$

For the heavy quarks, we find it more convenient to reexpress the \overline{m} in terms of the *pole* masses m : one has [8]

$$\begin{aligned}
\overline{m}(t) &= m \left[\frac{\alpha_s(t)}{\alpha_s(m^2)} \right]^{-\gamma_0/\beta_0} \left\{ 1 + \frac{A \alpha_s(t) - (C_F - A) \alpha_s(m^2)}{\pi} \right\} , \\
A \equiv \frac{\beta_1 \gamma_0 - \beta_0 \gamma_1}{\beta_0^2} \quad , \quad \gamma_0 &= -4 \quad , \quad \beta_1 = 102 - \frac{38}{3} n_f \quad , \quad \gamma_1 = -\frac{202}{3} + \frac{20}{9} n_f \quad ,
\end{aligned} \tag{4.8}$$

with the values of the pole masses given by [9]

$$m_c = 1570 \pm 60 \text{ MeV} \quad , \quad m_b = 4906 \pm 85 \text{ MeV} \quad , \tag{4.9}$$

and we take ^[10]

$$m_t = 180 \pm 12 \text{ GeV} \quad . \quad (4.10)$$

A crude estimate can now be obtained if one uses for $R_q(t)$ ($q = c, b, t$) the following approximation:

$$R_q^{(I)}(t) = \begin{cases} R_q^{\text{l.e.}} & \text{if } v_{crit} < v \leq 1/2 \\ R_q^{\text{h.e.}} & \text{if } v > 1/2 \end{cases} \quad (4.11)$$

The expression is however discontinuous at $v = 1/2$. To obtain a smooth joining of the low and high energy regions, we can use the following two interpolation curves for $R_q(t)$:

$$R_q^{(II)}(t) = (1 - v)R_q^{\text{l.e.}} + v R_q^{\text{h.e.}} \quad , \quad (4.12)$$

$$R_q^{(III)}(t) = (1 - v^3)R_q^{\text{l.e.}} + v^3 R_q^{\text{h.e.}} \quad . \quad (4.13)$$

We consider $R_q^{(II)}$ to be a very reasonable approximation, and take the difference between $R_q^{(I,II,III)}$ as an estimate of the systematic error due interpolations. We also take into account the error due to the input parameters $\alpha_s(m_\tau)$, $\alpha_s(M_Z)$, and m_t , as well as the error due to the non-perturbative contribution. The last is obtained by setting $\langle \alpha_s G^2 \rangle = 0$ in Eq. (4.6) and allowing the velocity v of the heavy quarks to go to zero. Our best estimate for the contribution of the regions in the continuum is:

$$a_h(\text{continuum}, t^{1/2} > \sqrt{2} \text{ GeV}) = (826.4 \pm 7.8 \pm 9.8 \pm 8) \times 10^{-11} \quad , \quad (4.14)$$

$$\Delta a_h(\text{continuum}, t^{1/2} > \sqrt{2} \text{ GeV}) = (226.6 \pm 3.7 \pm 1 \pm 1) \times 10^{-4} \quad . \quad (4.15)$$

The first systematic error is due to interpolations, the second is due to the error in the input parameters, and the last is the non-perturbative one.

V. CONCLUSION

Besides using our theoretical estimates for the high energy regions

$$t^{1/2} > 1.1 \text{ GeV}, \sqrt{2} \text{ GeV} \quad (5.1)$$

for $\text{Im}\Pi(t)$ an extra (slight) improvement may be incorporated if we repeat the dispersive analysis of Ref. [2] in the ρ region using recent data. The reason for this improvement is the following: in that paper the ρ contribution to (say) $a_h(t^{1/2} < 0.9 \text{ GeV})$, had been estimated from fits to the pion form factor F_π alone (but both in the spacelike and timelike regions) or imposing also the values of the $\pi\pi$ phase shifts. The first method gave $m_\rho = 768 \text{ MeV}$, the second $m_\rho = 778 \pm 2 \text{ MeV}$, both compatible (at the 2σ level) with the then preferred experimental value $m_\rho = 769 \pm 3 \text{ MeV}$. Although, as explained in Ref. [2], the first method was considered more reliable, both were combined taking the difference between the two determinations as a measure of the *systematic* error of the calculation. Since the presently accepted experimental figure, $m_\rho = 768.1 \pm 0.5 \text{ MeV}$ clearly discriminates in favor of the method based on F_π only, we can dispense with the (poorly known) $\pi\pi$ phase shifts and avoid the systematic error. We need only alter the evaluation by taking into account the change in the accepted value for the ρ width, $\Gamma_\rho = 151.5 \pm 1.2 \text{ MeV}$ from that used in Ref. [2], 158 MeV (compatible with the 1985 experimental value, $\Gamma_\rho = 154 \pm 5 \text{ MeV}$ but not with the presently preferred one). To first order this is easily taken into account as a variation $\Delta\Gamma/\pi m_\rho = -0.27\%$.

The results are summarized in the following tables where we also report, for purposes of comparison, the evaluations of the first (EJ) and third (MZ) papers of Ref. [1]. We have composed quadratically statistical and systematic errors: there are so many of the last, and of such varied origins, that they may be taken to behave statistically on the average. For the muon anomaly we have,

$$a_h(t^{1/2} < \sqrt{2}) = 6200.0 \pm 101.4 \times 10^{-11} \quad (\text{This work})$$

$$a_h(t^{1/2} < \sqrt{2}) = 6342.3 \pm 137.7 \times 10^{-11} \quad (\text{EJ, Ref. [1]})$$

and

$$a_h(t^{1/2} > \sqrt{2}) = 913.4 \pm 14.9 \times 10^{-11} \quad (\text{This work})$$

$$a_h(t^{1/2} > \sqrt{2}) = 908.2 \pm 77.1 \times 10^{-11} \quad (\text{EJ, Ref. [1]}) ,$$

with the overall results

$$a_h = 7113.4 \pm 102.5 \times 10^{-11} \quad (\text{This work})$$

$$a_h = 7250.4 \pm 157.6 \times 10^{-11} \quad (\text{EJ, Ref. [1]}).$$

which agree within the quoted errors.

For the running QED coupling, one has

$$\Delta\alpha_h(t^{1/2} < \sqrt{2}) = 50.3 \pm 1.6 \times 10^{-4} \quad (\text{This work})$$

$$\Delta\alpha_h(t^{1/2} < \sqrt{2}) = 47.92 \pm 1.06 \times 10^{-4} \quad (\text{EJ, Ref. [1]}),$$

and

$$\Delta\alpha_h(t^{1/2} > \sqrt{2}) = 239.12 \pm 4.1 \times 10^{-4} \quad (\text{This work})$$

$$\Delta\alpha_h(t^{1/2} > \sqrt{2}) = 232.45 \pm 6.45 \times 10^{-4} \quad (\text{EJ, Ref. [1]}),$$

and now the full results are

$$\Delta\alpha_h = 289.42 \pm 4.35 \times 10^{-4} \quad (\text{This work})$$

$$\Delta\alpha_h = 280.37 \pm 6.54 \times 10^{-4} \quad (\text{EJ, Ref. [1]}).$$

Our central value for $\Delta\alpha_h$ is slightly higher than that of Ref. [1] (EJ), but deviates at the 3-4 σ -level from that given by the third article of Ref. [1]:

$$\Delta\alpha_h = 273.2 \pm 4.2 \times 10^{-4} \quad (\text{MZ, Ref. [1]}).$$

We conclude this paper with a few comments on possible ways to improve the results. Certainly, better knowledge of α_s both on the τ and Z masses would increase the precision of the high energy evaluations; but most of the error comes from the region $0.8 \leq t^{1/2} \leq \sqrt{2}$ GeV. Even a modest improvement of a factor two in the error in the region between 0.8 and 1.1 GeV would result in a substantial decrease, roughly by the same amount, of the overall error for a_h , and about 20% for $\Delta\alpha_{\text{QED}}$. This emphasizes the interest of some of the accelerators, projected or in construction, with the capability to explore these energy ranges.

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